Acta Cryst. (1957). **10**, 121

Equations for Transforming Elastic and Piezoelectric Constants of Crystals*

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(*Received* 3 *July* 1956)

With the aid of tensor notation and methods, the equations for transforming the stiffness and compliance constants of crystals from one set of orthogonal axes to another are presented in compact tabular form. A similar treatment is also given for the piezoelectric constants.

1. Elastic constants

In full tensor notation, the generalized Hooke's law can be written (Mason, 1946, 1947; Wooster, 1938; Institution of Radio Engineers, 1949)

$$x_{ij} = s_{ijkl}X_{kl}\ , \qquad (1)$$

where x_{ij} are the strain components,
s_{ijkl} are the 'compliance constants',
X_{kl} are the stress components.

The suffixes i, j, k, l may take any values from 1 to 3, and the usual convention is observed whereby repetition of a suffix implies summation with respect to that suffix. Under a change of axes from x, y, z to x', y', z', where

$$\left.\begin{aligned} x' &= \alpha_1x+\alpha_2y+\alpha_3z\ , \\ y' &= \beta_1x+\beta_2y+\beta_3z\ , \\ z' &= \gamma_1x+\gamma_2y+\gamma_3z\ , \end{aligned}\right\} \qquad (2)$$

the compliance constants are transformed according to the equation

$$s'_{ijkl} = i_mj_nk_ol_ps_{mnop}\ , \qquad (3)$$

where i, j, k, l on the right-hand side now represent α, β or γ, writing 1 for α, 2 for β and 3 for γ, and m, n, o, p take the values 1, 2, or 3.

Equation (1) is almost invariably contracted to

$$x_q = s_{qr}X_r\ , \qquad (4)$$

where q and r may take values from 1 to 6, by replacing the suffixes

11 by 1, 22 by 2, 33 by 3, 23 by 4, 13 by 5 and 12 by 6. (5)

The stresses X_{kl} and X_r are true tensor components and X_{11} can be identified with X_1, X_{22} with X_2, X_{33} with X_3, X_{23} with X_4, X_{13} with X_5, and X_{12} with X_6. The strains x_q, as usually defined (Love, 1927), are not true tensor components and in order to conform with Voigt's definitions (1910) of s_{qr}, it is necessary to identify x_{11} with x_1, x_{22} with x_2, x_{33} with x_3, x_{23} with $\frac{1}{2}x_4$, x_{13} with $\frac{1}{2}x_5$ and x_{12} with $\frac{1}{2}x_6$. In changing over from equation (1) to equation (4), this introduces the numerical factors $\frac{1}{2}$ and $\frac{1}{4}$ as follows:

If $i = j,\ k = l\ (q, r = 1, 2, 3)$, $\quad s_{ijkl} = s_{qr}$.
If $i = j,\ k \neq l\ (q = 1, 2, 3,\ r = 4, 5, 6)$, $\quad s_{ijkl} = \frac{1}{2}s_{qr}$.
If $i \neq j,\ k \neq l\ (q, r = 4, 5, 6)$, $\quad s_{ijkl} = \frac{1}{4}s_{qr}$.

In terms of the 'stiffness constants' c_{ijkl}, the generalized Hooke's law (1) becomes

$$X_{ij} = c_{ijkl}x_{kl}\ , \qquad (6)$$

or, in contracted form,

$$X_q = c_{qr}x_r\ . \qquad (7)$$

The numerical factors $\frac{1}{2}$ and $\frac{1}{4}$ do not appear in changing over from equation (6) to equation (7), and $c_{ijkl} = c_{qr}$ for all values of q and r. The equation for transforming from the axes x, y, z to x', y', z' is analogous to (3):

$$c'_{ijkl} = i_mj_nk_ol_pc_{mnop}\ . \qquad (8)$$

In performing the summations indicated by equations (3) and (8), it should be borne in mind that as a result of the relations

$$x_{ij} = x_{ji},\ X_{kl} = X_{lk},\ s_{qr} = s_{rq},\ c_{qr} = c_{rq}\ ,$$

a given s_{ijkl} or c_{ijkl} can usually be written in more than one way, each way giving a separate term in the summation. For example, c_{1123}, c_{1132}, c_{2311} and c_{3211} are all equivalent to c_{14}, and the equation for c'_{qr} will therefore contain 4 terms in c_{14}. The cases which arise are those corresponding to the following combinations of suffixes:

$i\,i\,i\,i$	(1 term)
$i\,i\,j\,j$	(2 terms)
$i\,i\,i\,j$	(4 terms)
$i\,i\,j\,k$	(4 terms)
$i\,j\,i\,j$	(4 terms)
$i\,j\,j\,k$	(8 terms)

The developed equations (3) and (8) are given for

* This note was originally prepared in 1947, but has remained unpublished since. It is thought that its publication at the present time may be of interest in view of the recent appearance of the paper by Lieberman & Zirinsky (1956); see also Bechmann (1951).

Table 1. *Transformation equations of typical elastic constants*

	c'_{11}	c'_{12}	c'_{14}	c'_{15}	c'_{56}	c'_{66}
	(s'_{11})	(s'_{12})	$\left(\frac{s'_{14}}{2}\right)$	$\left(\frac{s'_{15}}{2}\right)$	$\left(\frac{s'_{56}}{4}\right)$	$\left(\frac{s'_{66}}{4}\right)$
$c_{11}\ (s_{11})$	α_1^4	$\alpha_1^2\beta_1^2$	$\alpha_1^2\beta_1\gamma_1$	$\alpha_1^3\gamma_1$	$\alpha_1^2\beta_1\gamma_1$	$\alpha_1^2\beta_1^2$
$c_{12}\ (s_{12})$	$2\alpha_1^2\alpha_2^2$	$\alpha_1^2\beta_2^2+\alpha_2^2\beta_1^2$	$\alpha_1^2\beta_2\gamma_2+\alpha_2^2\beta_1\gamma_1$	$\alpha_1\alpha_2^2\gamma_1+\alpha_1^2\alpha_2\gamma_2$	$\alpha_1\alpha_2\beta_2\gamma_1+\alpha_1\alpha_2\beta_1\gamma_2$	$2\alpha_1\alpha_2\beta_1\beta_2$
$c_{13}\ (s_{13})$	$2\alpha_1^2\alpha_3^2$	$\alpha_1^2\beta_3^2+\alpha_3^2\beta_1^2$	$\alpha_1^2\beta_3\gamma_3+\alpha_3^2\beta_1\gamma_1$	$\alpha_1\alpha_3^2\gamma_1+\alpha_1^2\alpha_3\gamma_3$	$\alpha_1\alpha_3\beta_3\gamma_1+\alpha_1\alpha_3\beta_1\gamma_3$	$2\alpha_1\alpha_3\beta_1\beta_3$
$c_{14}\left(\frac{s_{14}}{2}\right)$	$4\alpha_1^2\alpha_2\alpha_3$	$2\alpha_1^2\beta_2\beta_3+2\alpha_2\alpha_3\beta_1^2$	$\alpha_1^2\beta_3\gamma_2+\alpha_1^2\beta_2\gamma_3$ $+2\alpha_2\alpha_3\beta_1\gamma_1$	$\alpha_1^2\alpha_2\gamma_3+\alpha_1^2\alpha_3\gamma_2$ $+2\alpha_1\alpha_2\alpha_3\gamma_1$	$\alpha_1\alpha_2\beta_3\gamma_1+\alpha_1\alpha_3\beta_2\gamma_1$ $+\alpha_1\alpha_3\beta_1\gamma_2+\alpha_1\alpha_2\beta_1\gamma_3$	$2\alpha_1\alpha_3\beta_1\beta_2+2\alpha_1\alpha_2\beta_1\beta_3$
$c_{15}\left(\frac{s_{15}}{2}\right)$	$4\alpha_1^3\alpha_3$	$2\alpha_1^2\beta_1\beta_3+2\alpha_1\alpha_3\beta_1^2$	$\alpha_1^2\beta_1\gamma_3+\alpha_1^2\beta_3\gamma_1$ $+2\alpha_1\alpha_3\beta_1\gamma_1$	$3\alpha_1^2\alpha_3\gamma_1+\alpha_1^3\gamma_3$	$\alpha_1^2\beta_3\gamma_1+\alpha_1^2\beta_1\gamma_3+2\alpha_1\alpha_3\beta_1\gamma_1$	$2\alpha_1^2\beta_1\beta_3+2\alpha_1\alpha_3\beta_1^2$
$c_{16}\left(\frac{s_{16}}{2}\right)$	$4\alpha_1^3\alpha_2$	$2\alpha_1^2\beta_1\beta_2+2\alpha_1\alpha_2\beta_1^2$	$\alpha_1^2\beta_1\gamma_2+\alpha_1^2\beta_2\gamma_1$ $+2\alpha_1\alpha_2\beta_1\gamma_1$	$3\alpha_1^2\alpha_2\gamma_1+\alpha_1^3\gamma_2$	$\alpha_1^2\beta_2\gamma_1+\alpha_1^2\beta_1\gamma_2+2\alpha_1\alpha_2\beta_1\gamma_1$	$2\alpha_1^2\beta_1\beta_2+2\alpha_1\alpha_2\beta_1^2$
$c_{22}\ (s_{22})$	α_2^4	$\alpha_2^2\beta_2^2$	$\alpha_2^2\beta_2\gamma_2$	$\alpha_2^3\gamma_2$	$\alpha_2^2\beta_2\gamma_2$	$\alpha_2^2\beta_2^2$
$c_{23}\ (s_{23})$	$2\alpha_2^2\alpha_3^2$	$\alpha_2^2\beta_3^2+\alpha_3^2\beta_2^2$	$\alpha_2^2\beta_3\gamma_3+\alpha_3^2\beta_2\gamma_2$	$\alpha_2^2\alpha_3\gamma_3+\alpha_2\alpha_3^2\gamma_2$	$\alpha_2\alpha_3\beta_3\gamma_2+\alpha_2\alpha_3\beta_2\gamma_3$	$2\alpha_2\alpha_3\beta_2\beta_3$
$c_{24}\left(\frac{s_{24}}{2}\right)$	$4\alpha_2^3\alpha_3$	$2\alpha_2^2\beta_2\beta_3+2\alpha_2\alpha_3\beta_2^2$	$\alpha_2^2\beta_2\gamma_3+\alpha_2^2\beta_3\gamma_2$ $+2\alpha_2\alpha_3\beta_2\gamma_2$	$3\alpha_2^2\alpha_3\gamma_2+\alpha_2^3\gamma_3$	$\alpha_2^2\beta_3\gamma_2+\alpha_2^2\beta_2\gamma_3+2\alpha_2\alpha_3\beta_2\gamma_2$	$2\alpha_2^2\beta_2\beta_3+2\alpha_2\alpha_3\beta_2^2$
$c_{25}\left(\frac{s_{25}}{2}\right)$	$4\alpha_1\alpha_2^2\alpha_3$	$2\alpha_2^2\beta_1\beta_3+2\alpha_1\alpha_3\beta_2^2$	$\alpha_2^2\beta_3\gamma_1+\alpha_2^2\beta_1\gamma_3$ $+2\alpha_1\alpha_3\beta_2\gamma_2$	$\alpha_2^2\alpha_3\gamma_1+\alpha_1\alpha_2^2\gamma_3$ $+2\alpha_1\alpha_2\alpha_3\gamma_2$	$\alpha_1\alpha_2\beta_3\gamma_2+\alpha_2\alpha_3\beta_1\gamma_2$ $+\alpha_2\alpha_3\beta_2\gamma_1+\alpha_1\alpha_2\beta_2\gamma_3$	$2\alpha_1\alpha_2\beta_2\beta_3+2\alpha_2\alpha_3\beta_1\beta_2$
$c_{26}\left(\frac{s_{26}}{2}\right)$	$4\alpha_1\alpha_2^3$	$2\alpha_2^2\beta_1\beta_2+2\alpha_1\alpha_2\beta_2^2$	$\alpha_2^2\beta_1\gamma_2+\alpha_2^2\beta_2\gamma_1$ $+2\alpha_1\alpha_2\beta_2\gamma_2$	$3\alpha_1\alpha_2^2\gamma_2+\alpha_2^3\gamma_1$	$\alpha_2^2\beta_1\gamma_2+\alpha_2^2\beta_2\gamma_1+2\alpha_1\alpha_2\beta_2\gamma_2$	$2\alpha_1\alpha_2\beta_2^2+2\alpha_2^2\beta_1\beta_2$
$c_{33}\ (s_{33})$	α_3^4	$\alpha_3^2\beta_3^2$	$\alpha_3^2\beta_3\gamma_3$	$\alpha_3^3\gamma_3$	$\alpha_3^2\beta_3\gamma_3$	$\alpha_3^2\beta_3^2$
$c_{34}\left(\frac{s_{34}}{2}\right)$	$4\alpha_2\alpha_3^3$	$2\alpha_3^2\beta_2\beta_3+2\alpha_2\alpha_3\beta_3^2$	$\alpha_3^2\beta_2\gamma_3+\alpha_3^2\beta_3\gamma_2$ $+2\alpha_2\alpha_3\beta_3\gamma_3$	$3\alpha_2\alpha_3^2\gamma_3+\alpha_3^3\gamma_2$	$\alpha_3^2\beta_2\gamma_3+\alpha_3^2\beta_3\gamma_2+2\alpha_2\alpha_3\beta_3\gamma_3$	$2\alpha_2\alpha_3\beta_3^2+2\alpha_3^2\beta_2\beta_3$
$c_{35}\left(\frac{s_{35}}{2}\right)$	$4\alpha_1\alpha_3^3$	$2\alpha_3^2\beta_1\beta_3+2\alpha_1\alpha_3\beta_3^2$	$\alpha_3^2\beta_1\gamma_3+\alpha_3^2\beta_3\gamma_1$ $+2\alpha_1\alpha_3\beta_3\gamma_3$	$3\alpha_1\alpha_3^2\gamma_3+\alpha_3^3\gamma_1$	$\alpha_3^2\beta_1\gamma_3+\alpha_3^2\beta_3\gamma_1+2\alpha_1\alpha_3\beta_3\gamma_3$	$2\alpha_1\alpha_3\beta_3^2+2\alpha_3^2\beta_1\beta_3$
$c_{36}\left(\frac{s_{36}}{2}\right)$	$4\alpha_1\alpha_2\alpha_3^2$	$2\alpha_3^2\beta_1\beta_2+2\alpha_1\alpha_2\beta_3^2$	$\alpha_3^2\beta_2\gamma_1+\alpha_3^2\beta_1\gamma_2$ $+2\alpha_1\alpha_2\beta_3\gamma_3$	$\alpha_1\alpha_3^2\gamma_2+\alpha_2\alpha_3^2\gamma_1$ $+2\alpha_1\alpha_2\alpha_3\gamma_3$	$\alpha_1\alpha_3\beta_2\gamma_3+\alpha_2\alpha_3\beta_1\gamma_3$ $+\alpha_2\alpha_3\beta_3\gamma_1+\alpha_1\alpha_3\beta_3\gamma_2$	$2\alpha_1\alpha_3\beta_2\beta_3+2\alpha_2\alpha_3\beta_1\beta_3$
$c_{44}\left(\frac{s_{44}}{4}\right)$	$4\alpha_2^2\alpha_3^2$	$4\alpha_2\alpha_3\beta_2\beta_3$	$2\alpha_2\alpha_3\beta_2\gamma_3+2\alpha_2\alpha_3\beta_3\gamma_2$	$2\alpha_2^2\alpha_3\gamma_3+2\alpha_2\alpha_3^2\gamma_2$	$\alpha_2^2\beta_3\gamma_3+\alpha_3^2\beta_2\gamma_2$ $+\alpha_2\alpha_3\beta_2\gamma_3+\alpha_2\alpha_3\beta_3\gamma_2$	$\alpha_2^2\beta_3^2+\alpha_3^2\beta_2^2$ $+2\alpha_2\alpha_3\beta_2\beta_3$
$c_{45}\left(\frac{s_{45}}{4}\right)$	$8\alpha_1 x_2\alpha_3^2$	$4\alpha_2\alpha_3\beta_1\beta_3+4\alpha_1\alpha_3\beta_2\beta_3$	$2\alpha_2\alpha_3\beta_1\gamma_3+2\alpha_2\alpha_3\beta_3\gamma_1$ $+2\alpha_1\alpha_3\beta_2\gamma_3+2\alpha_1\alpha_3\beta_3\gamma_2$	$2\alpha_1\alpha_3^2\gamma_2+2\alpha_2\alpha_3^2\gamma_1$ $+4\alpha_1\alpha_2\alpha_3\gamma_3$	$\alpha_3^2\beta_1\gamma_2+\alpha_3^2\beta_2\gamma_1+\alpha_2\alpha_3\beta_1\gamma_3+\alpha_1\alpha_3\beta_3\gamma_2$ $+\alpha_2\alpha_3\beta_3\gamma_1+\alpha_1\alpha_3\beta_2\gamma_3+2\alpha_1\alpha_2\beta_3\gamma_3$	$2\alpha_1\alpha_2\beta_3^2+2\alpha_3^2\beta_1\beta_2$ $+2\alpha_1\alpha_3\beta_2\beta_3+2\alpha_2\alpha_3\beta_1\beta_3$
$c_{46}\left(\frac{s_{46}}{4}\right)$	$8\alpha_1\alpha_2^2\alpha_3$	$4\alpha_2\alpha_3\beta_1\beta_2+4\alpha_1\alpha_2\beta_2\beta_3$	$2\alpha_2\alpha_3\beta_1\gamma_2+2\alpha_2\alpha_3\beta_2\gamma_1$ $+2\alpha_1\alpha_2\beta_2\gamma_3+2\alpha_1\alpha_2\beta_3\gamma_2$	$2\alpha_2^2\alpha_3\gamma_1+2\alpha_1\alpha_2^2\gamma_3$ $+4\alpha_1\alpha_2\alpha_3\gamma_2$	$\alpha_2^2\beta_1\gamma_3+\alpha_2^2\beta_3\gamma_1+\alpha_1\alpha_2\beta_2\gamma_3+\alpha_2\alpha_3\beta_1\gamma_2$ $+\alpha_1\alpha_2\beta_3\gamma_2+\alpha_2\alpha_3\beta_2\gamma_1+2\alpha_1\alpha_3\beta_2\gamma_2$	$2\alpha_1\alpha_3\beta_2^2+2\alpha_2^2\beta_1\beta_3$ $+2\alpha_1\alpha_2\beta_2\beta_3+2\alpha_2\alpha_3\beta_1\beta_2$
$c_{55}\left(\frac{s_{55}}{4}\right)$	$4\alpha_1^2\alpha_3^2$	$4\alpha_1\alpha_3\beta_1\beta_3$	$2\alpha_1\alpha_3\beta_1\gamma_3+2\alpha_1\alpha_3\beta_3\gamma_1$	$2\alpha_1\alpha_3^2\gamma_1+2\alpha_1^2\alpha_3\gamma_3$	$\alpha_1^2\beta_3\gamma_3+\alpha_3^2\beta_1\gamma_1$ $+\alpha_1\alpha_3\beta_1\gamma_3+\alpha_1\alpha_3\beta_3\gamma_1$	$\alpha_1^2\beta_3^2+\alpha_3^2\beta_1^2$ $+2\alpha_1\alpha_3\beta_1\beta_3$
$c_{56}\left(\frac{s_{56}}{4}\right)$	$8\alpha_1^2\alpha_2\alpha_3$	$4\alpha_1\alpha_3\beta_1\beta_2+4\alpha_1\alpha_2\beta_1\beta_3$	$2\alpha_1\alpha_3\beta_1\gamma_2+2\alpha_1\alpha_3\beta_2\gamma_1$ $+2\alpha_1\alpha_2\beta_1\gamma_3+2\alpha_1\alpha_2\beta_3\gamma_1$	$2\alpha_1^2\alpha_3\gamma_2+2\alpha_1^2\alpha_2\gamma_3$ $+4\alpha_1\alpha_2\alpha_3\gamma_1$	$\alpha_1^2\beta_2\gamma_3+\alpha_1^2\beta_3\gamma_2+\alpha_1\alpha_3\beta_2\gamma_1+\alpha_1\alpha_2\beta_3\gamma_1$ $+\alpha_1\alpha_2\beta_1\gamma_3+\alpha_1\alpha_3\beta_1\gamma_2+2\alpha_2\alpha_3\beta_1\gamma_1$	$2\alpha_1^2\beta_2\beta_3+2\alpha_2\alpha_3\beta_1^2$ $+2\alpha_1\alpha_2\beta_1\beta_3+2\alpha_1\alpha_3\beta_1\beta_2$
$c_{66}\left(\frac{s_{66}}{4}\right)$	$4\alpha_1^2\alpha_2^2$	$4\alpha_1\alpha_2\beta_1\beta_2$	$2\alpha_1\alpha_2\beta_1\gamma_2+2\alpha_1\alpha_2\beta_2\gamma_1$	$2\alpha_1\alpha_2^2\gamma_1+2\alpha_1^2\alpha_2\gamma_2$	$\alpha_1^2\beta_2\gamma_2+\alpha_2^2\beta_1\gamma_1$ $+\alpha_1\alpha_2\beta_2\gamma_1+\alpha_1\alpha_2\beta_1\gamma_2$	$\alpha_1^2\beta_2^2+\alpha_2^2\beta_1^2$ $+2\alpha_1\alpha_2\beta_1\beta_2$

certain particular cases in Table 1. The equations are to be read downwards, e.g.,

$$c'_{11} = \alpha_1^4 c_{11}+\ldots+4\alpha_1^2\alpha_2\alpha_3 c_{14}+\ldots ,$$

$$\frac{s'_{14}}{2} = \alpha_1^2\beta_1\gamma_1 s_{11}+\ldots + (\alpha_1^2\beta_3\gamma_2+\alpha_1^2\beta_2\gamma_3+2\alpha_2\alpha_3\beta_1\gamma_1)\frac{s_{14}}{2}+\ldots .$$

The six cases selected for presentation in Table 1 are typical ones from which the remaining 15 equations may be derived by suitable interchange of α, β and γ. In practice, the derivations can be made by means of the rules given in Table 2.

Table 2. *Rules for deriving transformation equations of remaining elastic constants*

To derive	write	for	in the equation for
$c'_{22}(s'_{22})$	β	α	$c'_{11}(s'_{11})$
$c'_{33}(s'_{33})$	γ	α	$c'_{11}(s'_{11})$
$c'_{13}(s'_{13})$	γ	β	$c'_{12}(s'_{12})$
$c'_{23}(s'_{23})$	γ	α	$c'_{12}(s'_{12})$
$c'_{25}(s'_{25}/2)$	$\{\alpha, \beta\}$	$\{\beta, \alpha\}$	$c'_{14}(s'_{14}/2)$
$c'_{36}(s'_{36}/2)$	$\{\alpha, \gamma\}$	$\{\gamma, \alpha\}$	$c'_{14}(s'_{14}/2)$
$c'_{16}(s'_{16}/2)$	β	γ	$c'_{15}(s'_{15}/2)$
$c'_{24}(s'_{24}/2)$	β	α	$c'_{15}(s'_{15}/2)$
$c'_{26}(s'_{26}/2)$	$\{\beta, \alpha\}$	$\{\alpha, \gamma\}$	$c'_{15}(s'_{15}/2)$
$c'_{34}(s'_{34}/2)$	$\{\beta, \gamma\}$	$\{\gamma, \alpha\}$	$c'_{15}(s'_{15}/2)$
$c'_{35}(s'_{35}/2)$	$\{\alpha, \gamma\}$	$\{\gamma, \alpha\}$	$c'_{15}(s'_{15}/2)$
$c'_{45}(s'_{45}/4)$	$\{\alpha, \gamma\}$	$\{\gamma, \alpha\}$	$c'_{56}(s'_{56}/4)$
$c'_{46}(s'_{46}/4)$	$\{\alpha, \beta\}$	$\{\beta, \alpha\}$	$c'_{56}(s'_{56}/4)$
$c'_{44}(s'_{44}/4)$	γ	α	$c'_{66}(s'_{66}/4)$
$c'_{55}(s'_{55}/4)$	γ	β	$c'_{66}(s'_{66}/4)$

As an alternative to the use of equations (3) and (8), the method given by Voigt (1910, pp. 589, 595) may be used to obtain the developed equations; Voigt's method has in fact been used to check the equations in Table 1. Certain of these equations are well known. The full equations for s'_{34} and s'_{35} (which are simply related to the equation for s'_{15}) have been derived by Goens (1932). Voigt (1910) gives the key terms in the expressions for s'_{11}, s'_{33}, s'_{23}, s'_{44} and c'_{11}, and Cady (1946) gives those in the expressions for s'_{11}, s'_{23}, s'_{44}, c'_{11}, c'_{23} and c'_{44}, the remaining terms being obtained by cyclic interchange of suffixes.

The equations as they stand refer to the most general rotation in a material possessing the maximum number (21) of independent elastic constants. In a material which, owing to its symmetry, possesses a smaller number of independent elastic constants, the equations are simplified (see, for example, Hearmon, 1946). The equations are also simplified if the rotation is restricted so that one axis remains unaltered in position. For a rotation θ from x towards y about the axis of z, equations (2) become:

$$x' = x\cos\theta+y\sin\theta ,$$
$$y' = -x\sin\theta+y\cos\theta ,$$
$$z' = z ,$$

i.e.

$$\left.\begin{array}{lll} \alpha_1 = m, & \alpha_2 = n, & \alpha_3 = 0 , \\ \beta_1 = -n, & \beta_2 = m, & \beta_3 = 0 , \\ \gamma_1 = 0, & \gamma_2 = 0, & \gamma_3 = 1 , \end{array}\right\} \quad (9)$$

where $m = \cos\theta$, $n = \sin\theta$.

The system of equations for transforming the elastic compliances under this rotation was originally given by Voigt (1910, p. 593). The equations have been frequently quoted, but Cady (1946, p. 71) has pointed out that they contain some misprints. Substitution of the direction cosines (9) in the full set of equations corresponding with Table 1 shows, however, that there are two further misprints, one in the equation for s'_{12} and the other in the equation for s'_{36}. The correct equations are as quoted by Cady, except that

$$s'_{12} = (s_{11}+s_{22}-s_{66})m^2n^2+s_{12}(m^4+n^4) - (s_{16}-s_{26})(m^3n-mn^3) ,$$
$$s'_{36} = 2(s_{23}-s_{13})mn+s_{36}(m^2-n^2) .$$

2. Piezoelectric constants

The piezoelectric properties of crystals are expressed by the tensor equations*

$$P_i = e_{ijk}x_{jk} = d_{ijk}X_{jk} , \quad (10)$$

where P_i is the polarization, e and d are piezoelectric constants and moduli respectively, and i, j and k can take the values 1, 2 or 3. In the more usual contracted form, equations (10) become

$$P_i = e_{ir}x_r = d_{ir}X_r , \quad (11)$$

where r may have any value from 1 to 6, the actual value in terms of jk being determined by the rules (5) applicable to the elastic constants.

Owing to the fact that the x_r are not tensor components, the constants d_{ir} are expressed in terms of d_{ijk} as follows:

If $\quad j = k\ (r = 1, 2, 3),\ d_{ijk} = d_{ir}$.

If $\quad j \neq k\ (r = 4, 5, 6),\ d_{ijk} = \frac{1}{2}d_{ir}$.

The constants e_{ir} are given by $e_{ijk} = e_{ir}$, for all values of r.

The transformation from x, y, z to x', y', z' takes place according to the equations

* Equations (10) and (11) correspond to Voigt's formulation of piezoelectric theory. For ferroelectric materials, such as Rochelle salt, other formulations may be preferable; see Cady (1946, particularly chap. 11), Mason (1947, 1950), Haskins & Hickman (1950), and Bechmann (1953).

$$d'_{ijk} = i_m j_n k_o d_{mno} \,,$$
$$e'_{ijk} = i_m j_n k_o e_{mno} \,,$$

(cf. equations (3)), and the 4 typical developed equations are given in Table 3. The remaining 14 equations can be obtained by suitable interchange of α, β and γ, in practice by means of the rules given in Table 4. Voigt (1910, pp. 837, 839) has described an alternative method of deriving the developed equations, and the equations in Table 3 have been checked by Voigt's method.

Table 3. *Transformation equations of typical piezoelectric constants*

	e'_{11} (d'_{11})	e'_{12} (d'_{12})	e'_{14} ($d'_{14}/2$)	e'_{26} ($d'_{26}/2$)
$e_{11}(d_{11})$	α_1^3	$\alpha_1\beta_1^2$	$\alpha_1\beta_1\gamma_1$	$\alpha_1\beta_1^2$
$e_{12}(d_{12})$	$\alpha_1\alpha_2^2$	$\alpha_1\beta_2^2$	$\alpha_1\beta_2\gamma_2$	$\alpha_2\beta_1\beta_2$
$e_{13}(d_{13})$	$\alpha_1\alpha_3^2$	$\alpha_1\beta_3^2$	$\alpha_1\beta_3\gamma_3$	$\alpha_3\beta_1\beta_3$
$e_{14}(d_{14}/2)$	$2\alpha_1\alpha_2\alpha_3$	$2\alpha_1\beta_2\beta_3$	$\alpha_1\beta_2\gamma_3+\alpha_1\beta_3\gamma_2$	$\alpha_2\beta_1\beta_3+\alpha_3\beta_1\beta_2$
$e_{15}(d_{15}/2)$	$2\alpha_1^2\alpha_3$	$2\alpha_1\beta_1\beta_3$	$\alpha_1\beta_1\gamma_3+\alpha_1\beta_3\gamma_1$	$\alpha_1\beta_1\beta_3+\alpha_3\beta_1^2$
$e_{16}(d_{16}/2)$	$2\alpha_1^2\alpha_2$	$2\alpha_1\beta_1\beta_2$	$\alpha_1\beta_1\gamma_2+\alpha_1\beta_2\gamma_1$	$\alpha_1\beta_1\beta_2+\alpha_2\beta_1^2$
$e_{21}(d_{21})$	$\alpha_1^2\alpha_2$	$\alpha_2\beta_1^2$	$\alpha_2\beta_1\gamma_1$	$\alpha_1\beta_1\beta_2$
$e_{22}(d_{22})$	α_2^3	$\alpha_2\beta_2^2$	$\alpha_2\beta_2\gamma_2$	$\alpha_2\beta_2^2$
$e_{23}(d_{23})$	$\alpha_2\alpha_3^2$	$\alpha_2\beta_3^2$	$\alpha_2\beta_3\gamma_3$	$\alpha_3\beta_2\beta_3$
$e_{24}(d_{24}/2)$	$2\alpha_2^2\alpha_3$	$2\alpha_2\beta_2\beta_3$	$\alpha_2\beta_3\gamma_2+\alpha_2\beta_2\gamma_3$	$\alpha_2\beta_2\beta_3+\alpha_3\beta_2^2$
$e_{25}(d_{25}/2)$	$2\alpha_1\alpha_2\alpha_3$	$2\alpha_2\beta_1\beta_3$	$\alpha_2\beta_1\gamma_3+\alpha_2\beta_3\gamma_1$	$\alpha_1\beta_2\beta_3+\alpha_3\beta_1\beta_2$
$e_{26}(d_{26}/2)$	$2\alpha_1\alpha_2^2$	$2\alpha_2\beta_1\beta_2$	$\alpha_2\beta_1\gamma_2+\alpha_2\beta_2\gamma_1$	$\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2$
$e_{31}(d_{31})$	$\alpha_1^2\alpha_3$	$\alpha_3\beta_1^2$	$\alpha_3\beta_1\gamma_1$	$\alpha_1\beta_1\beta_3$
$e_{32}(d_{32})$	$\alpha_2^2\alpha_3$	$\alpha_3\beta_2^2$	$\alpha_3\beta_2\gamma_2$	$\alpha_2\beta_2\beta_3$
$e_{33}(d_{33})$	α_3^3	$\alpha_3\beta_3^2$	$\alpha_3\beta_3\gamma_3$	$\alpha_3\beta_3^2$
$e_{34}(d_{34}/2)$	$2\alpha_2\alpha_3^2$	$2\alpha_3\beta_2\beta_3$	$\alpha_3\beta_3\gamma_2+\alpha_3\beta_2\gamma_3$	$\alpha_3\beta_2\beta_3+\alpha_2\beta_3^2$
$e_{35}(d_{35}/2)$	$2\alpha_1\alpha_3^2$	$2\alpha_3\beta_1\beta_3$	$\alpha_3\beta_1\gamma_3+\alpha_3\beta_3\gamma_1$	$\alpha_3\beta_1\beta_3+\alpha_1\beta_3^2$
$e_{36}(d_{36}/2)$	$2\alpha_1\alpha_2\alpha_3$	$2\alpha_3\beta_1\beta_2$	$\alpha_3\beta_1\gamma_2+\alpha_3\beta_2\gamma_1$	$\alpha_1\beta_2\beta_3+\alpha_2\beta_1\beta_3$

As in the case of elastic constants, the equations are simplified (*a*) if the material possesses less than 18 independent piezoelectric constants, and (*b*) if the rotation is restricted so that one axis remains unaltered in position. Voigt (1910) has derived the equations for e_{ir} corresponding to the rotation (9) about the z axis and these equations have been quoted by Cady (1946). Insertion of the direction cosines (9) in the full set of equations corresponding with Table 3 leads exactly to the system of equations given by Voigt and Cady.

Table 4. *Rules for deriving transformation equations of remaining piezoelectric constants*

To derive	write	for	in the equation for
$e'_{22}(d'_{22})$	β	α	$e'_{11}(d'_{11})$
$e'_{33}(d'_{33})$	γ	α	$e'_{11}(d'_{11})$
$e'_{13}(d'_{13})$	γ	β	$e'_{12}(d'_{12})$
$e'_{21}(d'_{21})$	$\left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right.$	$\left.\begin{matrix}\beta\\ \alpha\end{matrix}\right\}$	$e'_{12}(d'_{12})$
$e'_{23}(d'_{23})$	$\left\{\begin{matrix}\gamma\\ \beta\end{matrix}\right.$	$\left.\begin{matrix}\beta\\ \alpha\end{matrix}\right\}$	$e'_{12}(d'_{12})$
$e'_{31}(d'_{31})$	$\left\{\begin{matrix}\alpha\\ \gamma\end{matrix}\right.$	$\left.\begin{matrix}\beta\\ \alpha\end{matrix}\right\}$	$e'_{12}(d'_{12})$
$e'_{32}(d'_{32})$	γ	α	$e'_{12}(d'_{12})$
$e'_{15}(d'_{15}/2)$	$\left\{\begin{matrix}\alpha\\ \gamma\end{matrix}\right.$	$\left.\begin{matrix}\beta\\ \alpha\end{matrix}\right\}$	$e'_{26}(d'_{26}/2)$
$e'_{16}(d'_{16}/2)$	$\left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right.$	$\left.\begin{matrix}\beta\\ \alpha\end{matrix}\right\}$	$e'_{26}(d'_{26}/2)$
$e'_{24}(d'_{24}/2)$	γ	α	$e'_{26}(d'_{26}/2)$
$e'_{34}(d'_{34}/2)$	$\left\{\begin{matrix}\beta\\ \gamma\end{matrix}\right.$	$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\}$	$e'_{26}(d'_{26}/2)$
$e'_{35}(d'_{35}/2)$	γ	β	$e'_{26}(d'_{26}/2)$
$e'_{25}(d'_{25}/2)$	$\left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right.$	$\left.\begin{matrix}\beta\\ \alpha\end{matrix}\right\}$	$e'_{14}(d'_{14}/2)$
$e'_{36}(d'_{36}/2)$	$\left\{\begin{matrix}\alpha\\ \gamma\end{matrix}\right.$	$\left.\begin{matrix}\gamma\\ \alpha\end{matrix}\right\}$	$e'_{14}(d'_{14}/2)$

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